Wind-Driven Ocean Circulation

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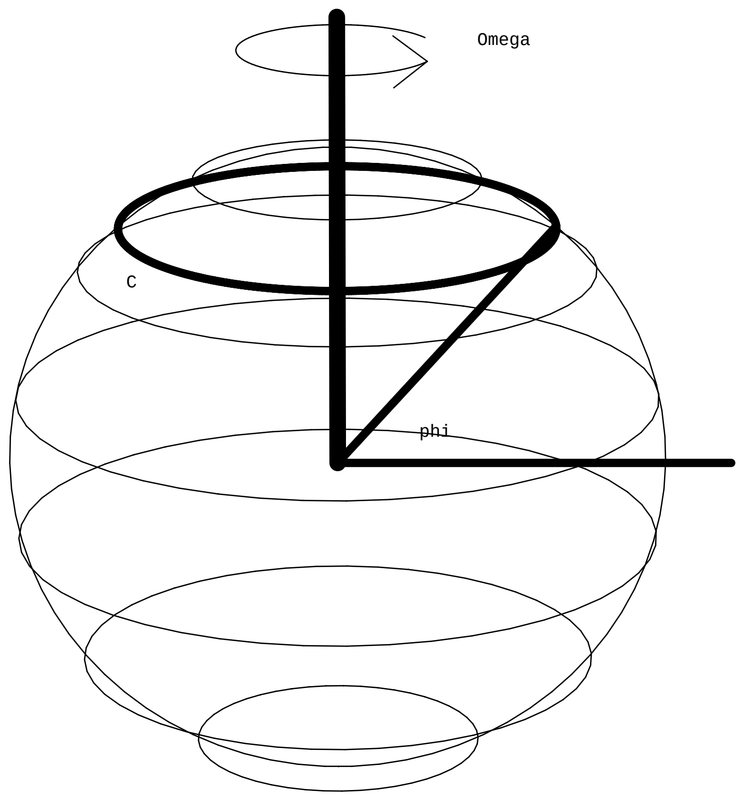
Omega
C
phi

Equations of Motion in a Rotating Frame

Consider a body rotating with angular velocity Ω about an axis $\mathbf{k}$. Let $\mathbf{\Omega} = \Omega\mathbf{k}$. A typical particle located at position $\mathbf{r}$ relative to the rotating frame will have a velocity

$$\mathbf{\Omega} \times \mathbf{r}$$

in an inertial frame. Let $\mathbf{v}$ be the velocity of a particle relative to a rotating body and $\mathbf{v}_i$ be its velocity in an inertial frame. Then

$$\mathbf{v}_i = \mathbf{v} + \mathbf{\Omega} \times \mathbf{r}.$$

Define

$$D_i = D_t + \mathbf{\Omega}\times$$

as the time-differentiation operator in an inertial frame. Then acceleration in an inertial frame takes the form (Note that $D_t\mathbf{r} = \mathbf{v}$)

$$D_i(D_i\mathbf{r}) = (D_t + \mathbf{\Omega}\times)(D_t\mathbf{r} + \mathbf{\Omega} \times \mathbf{r}) =$$

$$\mathbf{a} + 2\mathbf{\Omega} \times \mathbf{v} + \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r}.$$

The term $\mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{r}$, the centripetal acceleration, is about 0.03 meters per second square on our planet and will be neglected relative to g, which is about 9.8 meters per second square.

The term $2\mathbf{\Omega}\times\mathbf{v}$ is the **Coriolis** term.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis at a point P, $\mathbf{e}_1$ pointing eastward, $\mathbf{e}_2$ pointing northward, and $\mathbf{e}_3$ pointing in the radial direction. Let θ be the latitude at P. Then

$$\mathbf{\Omega} = \Omega \cos\theta \mathbf{e}_2 + \Omega \sin\theta \mathbf{e}_3.$$

Let u, v and w be the components of $\mathbf{v}$ in the basis:

$$\mathbf{v} = u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3.$$

Then $2\mathbf{\Omega} \times \mathbf{v}$ takes the form

$2\Omega(w \cos\theta - v \sin\theta)\mathbf{e}_1 + 2\Omega v \sin\theta \mathbf{e}_2 - u \cos\theta \mathbf{e}_3$.

In 2-D flows, the basic assumption is that w is small relative to u and v. With that in mind, and with f defined by

$$f = 2\Omega \sin\theta,$$

the Coriolis term is approximated by

$$2\mathbf{\Omega} \times \mathbf{v} = -fv\mathbf{e}_1 + fu\mathbf{e}_2.$$

In Fluid dynamics, velocity and acceleration are represented in terms of positions (as opposed to particles) – **Eulerian** representation. Thus

$$u = u(x, y, z, t).$$

Consider a fixed position (x, y, z).

Let $(X(t), Y(t), Z(t))$ be the set of particles that occupy (x, y, z) at each time t. Then

$$u = u(X(t), Y(t), Z(t), t)$$

and

$$a = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} X' + \frac{\partial u}{\partial y} Y' + \frac{\partial u}{\partial z} Z' =$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}.$$

The equations of **2-D flow** are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_1,$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_2,$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

George Veronis's Model (Journal of **the** Atmospheric Sciences, Vol 20, 1963, pp. 577 – 593)

$$f = f_0 + \beta y.$$

$$F_1 = -Ku + \frac{f_1}{H}, \quad F_2 = -Kv + \frac{f_2}{H}.$$

K is a frictional constant, $\langle f_1, f_2 \rangle$ models wind stress, and H is the depth of the basin.

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-(f_0+\beta y)v=-\frac{1}{\rho}\frac{\partial p}{\partial x}-Ku+\frac{f_1}{H},$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+(f_0+\beta y)u=-\frac{1}{\rho}\frac{\partial p}{\partial y}-Kv+\frac{f_2}{H}.$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

Cross-differentiate these equations and subtract to get rid of p.

Since $u_x+v_y=0$, let ψ be the stream function, i.e.,

$$u=-\frac{\partial \psi}{\partial y}, \quad v=\frac{\partial \psi}{\partial x}.$$

$$\Delta\psi_t + J(\psi, \Delta\psi) + \beta\frac{\partial\psi}{\partial x} = -K\Delta\psi + \frac{g}{H},$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and

$$J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}.$$

Initial-Boundary Value Problem

Solve the PDE with

$$g = -\frac{W}{L}\sin\frac{x}{L}\sin\frac{y}{L}, \quad \text{wind stress}$$

$$\psi(x,y,0) = \text{given}$$

and

$\psi = 0, \quad$ when $\quad x = y = 0, \quad$ and $\quad x = y = \pi L.$

Nondimensionalize variables:

$$x = Lx', \quad y = Ly' \quad t = \frac{t'}{L\beta}, \quad \psi = \frac{W}{\beta H}\psi'$$

New PDE:

$$\Delta\psi_t + RJ(\psi, \Delta\psi) + \frac{\partial\psi}{\partial x} = -\epsilon\Delta\psi - \sin x \sin y,$$

where

$$R = \frac{W}{\beta^2 H L^3}, \qquad \epsilon = \frac{K}{\beta L}.$$

Initial condition

$$\psi(x, y, 0) = \psi_0(x, y),$$

Boundary Condition:

$\psi(x, y, t) = 0$ on the boundary of $(0, \pi)\times(0, \pi)$

Pseudo-Spectral Method:

1. Choose a basis $\phi_{ij}(x,y)$ for $L^2((0,\pi)\times(0,\pi))$ which satisfy the boundary condition (say $\phi_{ij} = \sin ix \sin jy$ or $\phi_{ij} = T_i(x)T_j(y)$ where T_i is the Chebyshev polynomial on $(0,\pi)$).

2. Seek a solution of the form
$$\psi = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}(t)\phi_{ij}(x,y).$$

3. Substitute ψ into PDE, take inner product with ϕ_{mn}, $m = 1,2,...,N$, $n = 1,2,...,N$ and end up with N^2 **Ordinary** differential equations for a_{ij}. Solve this initial-value problem to get an approximate solution.

Let

$$L[\psi] = \Delta\psi_t + RJ(\psi, \Delta\psi) + \psi_x + \epsilon\Delta\psi + F$$

Then this **pseudo-spectral method** can be described by

$$(L[\psi], \phi_{mn}) = 0, \quad m = 1, ..., N, \quad n = 1, ..., N.$$

Example: Let $N = 2$ and let

$$\phi_{ij} = \sin ix \sin jy.$$

Then

$$\psi = a(t) \sin x \sin y + b(t) \sin 2x \sin y +$$

$$c(t) \sin x \sin 2y + d(t) \sin 2x \sin 2y$$

a, b, c, d satisfy the system:

$$a' = -\frac{4}{3\pi} b - \epsilon a + \frac{1}{2},$$

$$b' = \frac{8}{15\pi} a + \frac{9}{20} Rac - \epsilon b,$$

$$c' = -\frac{8}{15\pi} d - \frac{9}{20} Rab - \epsilon c,$$

$$d' = \frac{1}{3\pi} c - \epsilon d.$$

Equilibrium Solutions:

$$8192\,a-5760\,\epsilon\,\pi^2+57600\,a\,\epsilon^2\,\pi^2-32400\,\epsilon^3\,\pi^4+$$

$$64800\,a\,\epsilon^4\,\pi^4-6561\,a^2\,\epsilon\,\pi^4\,R^2+$$

$$13122\,a^3\,\epsilon^2\,\pi^4\,R^2=0$$